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A SURVEY ON GENERALIZED HERMITE CONSTANTS

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This is an expository note on Hermite's constant. We give an account of a recent development of some generalizations of Hermite's constant.

1. Hermite–Rankin's constant. Let $\mathcal{L}^n$ be the set of all lattices of rank n in the Euclidean space $\mathbb{R}^n$. For $L \in \mathcal{L}^n$, $d(L)$ stands for the volume of the fundamental parallelepiped of L. It was proved by Hermite that

$$\min_{0\neq x\in L} {}^t xx \leq \left(\frac{2}{\sqrt{3}}\right)^{n-1} d(L)^{2/n}$$

holds for all $L \in \mathcal{L}^n$. Thus $\min_{0\neq x\in L} {}^t xx/d(L)^{2/n}$ is bounded and there exists the maximum

$$\gamma_n = \max_{L\in\mathcal{L}^n} \min_{0\neq x\in L} \frac{{}^t xx}{d(L)^{2/n}}.$$

The constant γ_n is called Hermite's constant. A well-known example of its appearance is the lattice sphere packing problem, namely the density of the densest lattice packing of spheres in $\mathbb{R}^n$ equals

$$\delta_n = \gamma^{n/2} \frac{V(n)}{2^n},$$

where $V(n)$ denotes the volume of the unit ball in $\mathbb{R}^n$, i.e., $V(n) = \pi^{n/2}/\Gamma(1+n/2)$. Originally, γ_n arose from the reduction theory of positive definite quadratic forms initiated by Lagrange, Seeber and Gauss. In terms of quadratic forms, γ_n is represented as

$$\gamma_n = \max_{g\in GL_n(\mathbb{R})} \min_{0\neq x\in\mathbb{Z}^n} \frac{{}^t x {}^t g g x}{(\det g)^{2/n}}. \tag{1}$$

The exact value of γ_n is known only for $n \leq 8$, i.e., $\gamma_2 = 2/\sqrt{3}, \gamma_3 = \sqrt[3]{2}, \gamma_4 = \sqrt{2}, \gamma_5 = \sqrt[5]{8}, \gamma_6 = \sqrt[6]{64/3}, \gamma_7 = \sqrt[7]{64}, \gamma_8 = 2$. One has the estimate

$$\left(\frac{2\zeta(n)}{V(n)}\right)^{2/n} \leq \gamma_n \leq 4\left(\frac{1}{V(n)}\right)^{2/n}. \tag{2}$$

This upper bound was given by Minkowski and follows from $\delta_n \leq 1$. The lower bound was first stated by Minkowski and was proved by Hlawka.

The next step of Hermite's constant is the following extension due to Rankin. For every $1 \le d \le n-1$, define

$$(3) \qquad \gamma_{n,d} = \max_{L \in \mathcal{L}^n} \min_{\substack{x_1,\cdots,x_d \in L \\ x_1\wedge\cdots\wedge x_d \neq 0}} \frac{\det({}^t x_i x_j)_{1\le i,j\le d}}{d(L)^{2d/n}}.$$

Obviously, $\gamma_{n,1}$ equals γ_n. Rankin ([R]) proved $\gamma_{n,d}$ satisfies the inequality

$$\gamma_{n,d} \le \gamma_{m,d}(\gamma_{n,m})^{d/m}$$

for $1 \le d < m \le n-1$, and he showed $\gamma_{4,2} = 3/2$. Rankin's inequality and the duality $\gamma_{n,d} = \gamma_{n,n-d}$ yield Mordell's inequality $\gamma_n^{n-2} \le \gamma_{n-1}^{n-1}$.

2. Icaza–Thunder's generalization. As a generalization of Hermite–Rankin constant, Thunder defined the constant $\gamma_{n,d}(k)$ for any algebraic number field k of finite degree r over $\mathbb{Q}$ in 1997. At first, we recall a definition of twisted heights. Let $\mathbf{e}_1, \cdots, \mathbf{e}_n$ be a standard basis of k^n. For any extension field L over k, $W_{n,d}(L)$ stands for the d-th exterior product of L^n. A basis of $W_{n,d}(k)$ is formed by the elements $\mathbf{e}_I = \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_d}$ with $I = \{1 \le i_1 < i_2 < \cdots < i_d \le n\}$. For each place v of k, let k_v be the completion of k at v and $|\cdot|_v$ the usual normalized absolute value of k_v. We define the local height on $W_{n,d}(k_v)$ by

$$H_v(\sum_I a_I \mathbf{e}_I) = \begin{cases} \left(\sum_I |a_I|_v^{[\mathbb{C}:k_v]}\right)^{1/([\mathbb{C}:k_v]r)} & \text{(if } v \text{ is infinite)} \\ \left(\sup_I |a_I|_v\right)^{1/r} & \text{(if } v \text{ is finite)} \end{cases}$$

Then the global height H on $W_{n,d}(k)$ is defined to be the product of H_v:

$$H(x) = \prod_v H_v(x) \qquad (x \in W_{n,d}(k)).$$

Let $\mathbb{A}$ be the adele ring of k and $|\cdot|_{\mathbb{A}}$ the idele norm on $\mathbb{A}^\times$. Since $H(\alpha x) = |\alpha|_{\mathbb{A}}^{1/r} H(x) = H(x)$ for $\alpha \in k^\times$, H defines a height on the projective space $\mathbf{P}W_{n,d}(k)$. By the Plücker embedding, H is regarded as a height on the Grassmanian $\mathrm{Gr}_{n,d}(k)$ of all d-dimensional subspaces of k^n. For $X \in \mathrm{Gr}_{n,d}(k)$, $H(X)$ is precisely given by $H(x_1 \wedge \cdots \wedge x_d)$, where $x_1, \cdots, x_d$ is an arbitrary k-basis of X. More generally, for each $g = (g_v)$ in $GL_n(\mathbb{A})$, the twisted height H_g on $\mathrm{Gr}_{n,d}(k)$ is defined as

$$H_g(X) = \prod_v H_v(g_v x_1 \wedge \cdots \wedge g_v x_d).$$

Now the constant $\gamma_{n,d}(k)$ is defined to be

$$(4) \qquad \gamma_{n,d}(k) = \max_{g \in GL_n(\mathbb{A})} \min_{X \in \mathrm{Gr}_{n,d}(k)} \frac{H_g(X)^2}{|\det g|_{\mathbb{A}}^{2d/(nr)}}.$$

In the case of $k = \mathbb{Q}$, this definition is identical with (1) and (3), so that one has $\gamma_{n,d}(\mathbb{Q}) = \gamma_{n,d}$. As generalizations of Minkowski – Hlawka bound and Rankin's inequality, Thunder showed

Theorem. ([T]) *One has*

$$(5)\qquad \left(\frac{n|D_k|^{d(n-d)/2}}{\mathrm{Res}_{s=1}\zeta_k(s)} \frac{\prod\limits_{j=n-d+1}^{n} Z_k(j)}{\prod\limits_{j=2}^{d} Z_k(j)} \right)^{2/(nr)} \leq \gamma_{n,d}(k) \leq \left(\frac{2^{r_1+r_2}|D_k|^{1/2}}{V(n)^{r_1/n}V(2n)^{r_2/n}} \right)^{2d/r}$$

and

$$\gamma_{n,d}(k) \leq \gamma_{m,d}(k)(\gamma_{n,m}(k))^{d/m} \qquad (1 \leq d < m \leq n-1).$$

Here $Z_k(s) = (\pi^{-s/2}\Gamma(s/2))^{r_1}((2\pi)^{1-s}\Gamma(s))^{r_2}\zeta_k(s)$ *denotes the zeta function of* k, D_k *the discriminant of* k *and* r_1 *(resp.* r_2 *) the number of real (resp. imaginary) places of* k.

We particularly write $\gamma_n(k)$ for $\gamma_{n,1}(k)$. Newman ([N, XI]) and Icaza ([I]) also considered $\gamma_n(k)$ based on Humbert's reduction theory. Newman gave exact values of $\gamma_2(k)$ for some Eucledean imaginary quadratic fields. To be precise, one has $\gamma_2(\mathbb{Q}(\sqrt{-1})) = \sqrt{2}, \gamma_2(\mathbb{Q}(\sqrt{-2})) = 2, \gamma_2(\mathbb{Q}(\sqrt{-3})) = \sqrt{6}/2, \gamma_2(\mathbb{Q}(\sqrt{-7})) = \sqrt{21}/3$ and $\gamma_2(\mathbb{Q}(\sqrt{-11})) = \sqrt{22}/2$. As for $\gamma_2(k)$ of real quadratic fields, some numerical examples and conjectures were given by Cohn [C]. Recently, Coulangeon proved a part of Cohn's conjecture, i.e., $\gamma_2(\mathbb{Q}(\sqrt{2}) = 2/\sqrt{2\sqrt{6}-3}, \gamma_2(\mathbb{Q}(\sqrt{3})) = 4$ and $\gamma_2(\mathbb{Q}(\sqrt{5})) = 2/\sqrt[4]{5}$, by using the Voronoi reduction. In a general k, Ohno and the author obtained an upper bound of $\gamma_n(k)$ better than (5).

Theorem. ([O-W]) *One has*

$$\gamma_n(k) \leq |D_k|^{1/r}\frac{\gamma_{nr}(\mathbb{Q})}{r}.$$

Combining this with (5), *one obtains*

$$(6)\qquad \frac{r}{\pi}\left\{ \frac{nw_k\Gamma(n/2)^{r_1}\Gamma(n)^{r_2}\zeta_k(n)}{2^{r_1+nr_2}h_kR_k} \right\}^{2/(nr)} \leq \gamma_{nr}(\mathbb{Q})$$

for any algebraic number field k *of degree* r. *Here* h_k, R_k *and* w_k *denote the class number of* k, *the regulator of* k *and the number of the roots of unity in* k, *respectively.*

If a small n is fixed, there are some numerical examples that (6) for a suitable k is better than the Minkowski–Hlawka bound of $\gamma_{nr}(\mathbb{Q})$.

3. Generalized Hermite constants of flag varieties. Thunder's definition of Hermite's constant can be extended to flag varieties. In order to do this, we use a theory of linear algebraic groups. Let G be a connected reductive linear algebraic group defined over k and $\pi: G \to GL(V_\pi)$ a k-rational absolutely irreducible representation. We denote by D_π the highest weight line in V_π with respect to a fixed Borel subgroup of G. The stabilizer Q_π of D_π in G is a parabolic subgroup of G. The representation π is said to be strongly k-rational if Q_π is defined over k. Then the flag variety G/Q_π is defined over k and is embedded in the projective space $\mathbf{P}V_\pi$. Let $G(\mathbf{A})$ be the adele group of G and $G(\mathbf{A})^1$ the group consisting of $g \in G(\mathbf{A})$ such that $|\chi(g)|_{\mathbf{A}} = 1$ for any k-rational character χ of G. For each $g \in GL(V_\pi(\mathbf{A}))$, a twisted height H_g on $\mathbf{P}V_\pi(k)$ is defined similarly to §2. Then we can prove that the following maximum exists for any strongly k-rational π ([W, Proposition 2]):

$$\gamma_\pi^G = \max_{g\in G(\mathbf{A})^1} \min_{\gamma\in G(k)} H_{\pi(g\gamma)}(D_\pi)^2 ,$$

where we regard D_π as a k-rational point in $\mathbf{P}V_\pi$. If $G = GL_n$ and π is a d-th exterior representation π_d of G, then one sees $\gamma_{\pi_d}^{GL_n} = \gamma_{n,d}(k)$. A mean value argument used to prove Minkowski–Hlawka bound works well in this general setting (cf. [M-W, §3.3]).

Theorem. ([W]) *If $Q = Q_\pi$ is a maximal parabolic subgroup of G, we have a lower estimate of the form*

$$\left(\frac{C_Q d_G e_Q \tau(G)}{C_G d_Q \tau(Q)}\right)^{2e_\pi/(e_Q r)} \le \gamma_\pi^G . \tag{7}$$

Here $\tau(G)$ and $\tau(Q)$ denote the Tamagawa numbers of G and Q, respectively, d_G, d_Q, e_Q and e_π are some elementary positive rational numbers depending on G, Q and π, and furthermore C_G and C_Q are the volumes of some maximal compact subgroups of $G(\mathbf{A})$ and $Q(\mathbf{A})$, respectively.

If G is split over k, both constants C_G and C_Q are described by special values of the Dedekind zeta function. Particularly, the estimate (7) in the case of $G = GL_n$ and $\pi = \pi_d$ coincides with the lower bound of (5). An upper bound of γ_π^G is not yet known in general.

4. Some examples. We show two examples. First, let $F: k^n \times k^n \to k$ be a nondegenerate symmetric bilinear form of Witt index $q \ge 1$ and $G = SO_F$ be the special orthogonal group of F. For $1 \le d \le q$, the d-th exterior representation $\pi_d: G(k) \to GL(W_{n,d}(k))$ yields a strongly k-rational representation of G. (The case $q = n/2 = d$ is exceptional since π_q is not irreducible.) We write γ_d^F for the generalized Hermite constant $\gamma_{\pi_d}^G$. As an analogue of (4), γ_d^F has the following geometrical representation:

$$\gamma_d^F = \max_{g\in G(\mathbf{A})} \min_{X\in \mathrm{Gr}_{n,d}(k,F)} H_g(X)^2 ,$$

where $\mathrm{Gr}_{n,d}(k,F)$ denotes a subset of $\mathrm{Gr}_{n,d}(k)$ consisting of d-dimensional totally isotropic subspaces of k^n with respect to F. In particular, γ_1^F is related to an existence of a nontrivial small integral solution of the homogeneous quadratic equation $F(x,x)=0$. If $2q=n$ or $2q+1=n$, (7) gives

$$\gamma_1^F \geq \begin{cases} \left(\dfrac{|D_k|^{q-1}(2q-2)}{\mathrm{Res}_{s=1}\zeta_k(s)} \dfrac{Z_k(2(q-1))Z_k(q)}{Z_k(q-1)}\right)^{1/((q-1)r)} & (2q=n) \\ \left(\dfrac{|D_k|^{q-1/2}(2q-1)}{\mathrm{Res}_{s=1}\zeta_k(s)} Z_k(2q)\right)^{2/((2q-1)r)} & (2q+1=n) \end{cases}$$

Moreover, we can show the following estimate and an analogue of Rankin's inequality.

Theorem. ([O-W],[W2]) *For any nondegenerate F, one has*

$$\gamma_d^F \leq \gamma_{n-d}(k)^{n-d}(2H(F))^{n-d} \qquad (1 \leq d \leq q)$$
$$\gamma_d^F \leq \gamma_{m,d}(k)(\gamma_m^F)^{d/m} \qquad (1 \leq d < m \leq q).$$

Here $H(F)$ denotes a height of the symmetric matrix corresponding to F.

Second, let $\mathcal{D}$ be a central simple division algebra of dimension q^2 over k and G be an inner k-form of GL_{qn} whose group of k-rational points equals $GL_n(\mathcal{D})$. If a cyclic extension L of degree q over k contained in D is fixed, then $GL_n(\mathcal{D})$ is realized as a subgroup of $GL_{qn}(L)$. Since the qd-th exterior representation of $GL_{qn}(L)$ gives rise to a fundamental k-rational representation π_d of G for $1 \leq d \leq n-1$, one has the generalized Hermite constant $\gamma_{\pi_d}^G$. We write $\gamma_{n,d}(\mathcal{D})$ for $\gamma_{\pi_d}^G$. Geometrically, $\gamma_{n,d}(\mathcal{D})$ has the following representation similar to (4):

$$\gamma_{n,d}(\mathcal{D}) = \max_{g \in G(\mathbb{A}_k)} \min_{X \in \mathrm{BS}_{n,d}(\mathcal{D})} \frac{H_g(X)^2}{|\mathrm{Nr}(g)|^{2d/(nr)}},$$

where $\mathrm{BS}_{n,d}(\mathcal{D})$ denotes the set of d-dimensional $\mathcal{D}$-subspace in $\mathcal{D}^n$ and Nr the reduced norm on $M_n(\mathcal{D})$. The set $\mathrm{BS}_{n,d}(\mathcal{D})$ is called the generalized Brauer–Severi variety and is realized as a subset of the Grassmanian $\mathrm{Gr}_{qn,qd}(L)$. The twisted height H_g on $\mathrm{BS}_{n,d}(\mathcal{D})$ is defined as the restriction of that on $\mathrm{Gr}_{qn,qd}(L)$. By using this expression, we can prove the following.

Theorem. ([W3]) *One has*

$$\gamma_{n,d}(\mathcal{D}) \leq \epsilon_{\mathcal{D}} \left(\frac{2^{r_1(L)+r_2(L)}|D_L|^{1/2}}{V(qn)^{r_1(L)/(qn)}V(2qn)^{r_2(L)/(qn)}}\right)^{2d/r}$$

and

$$\gamma_{n,d}(\mathcal{D}) \leq \gamma_{m,d}(\mathcal{D})(\gamma_{n,m}(\mathcal{D}))^{d/m} \qquad (1 \leq d < m \leq n-1).$$

Here D_L denotes the discriminant of L and $r_1(L)$ (resp. $r_2(L)$) the number of real (resp. imaginary) places of L. The constant $\epsilon_{\mathcal{D}}$ is given by

$$\epsilon_{\mathcal{D}} = \left(\prod_w \max(1, |a|_w)\right)^{2(q-1)n/(qr)} \qquad (w \text{ runs over all places of } L)$$

if we realize $\mathcal{D}$ as a cyclic algebra $[L/k, \sigma, a]$ by a generator σ of the Galois group of L/k and an element $a \in k^\times$.

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